International Journal of Theoretical Physics, Vol. 37, No. 2, 1998

Consequences of the Noncompactness of the Lorentz group

Hans-Jürgen Schmidt[1]

Received July 1, 1997

The following four statements for indefinite metrics of Lorentz signature do not possess an analogous statement in the definite Euclidean signature case: (1) All curvature invariants of a gravitational wave vanish, in spite of the fact that it represents a nonflat spacetime. (2) The eigennullframe components of the curvature tensor (the Cartan "scalars") do not represent curvature scalars. (3) The Euclidean topology in the Minkowski spacetime does not possess a basis composed of Lorentz-invariant neighborhoods. (4) There are points in the de Sitter spacetime which cannot be joined to each other by any geodesic. We show that these four statements all follow from the noncompactness of the Lorentz group. We conclude that the popular (and often useful) imaginary-coordinate rotation from Euclidean to Lorentzian signature (called Wick rotation) is not an isomorphism.

1. INTRODUCTION

A topological space X is compact if each open cover contains a finite subcover. Equivalently one can say: X is compact if each sequence in X possesses a converging subsequence. $SO(n)$, the n-dimensional rotation group, is compact, whereas $SO(n - 1, 1)$, the corresponding Lorentz group, fails to be compact for $n \geq 2$. Nevertheless, one can simply switch from the Euclidean space E^n to the Minkowski spacetime M^n by replacing $x^n \to it$.

It is the aim of this paper to show those points where the loss of compactness connected with this replacement has nontrivial consequences.

2. GRAVITATIONAL WAVES

Let

$$ds^2 = 2\, du\, dv - a^2(u) dw^2 - b^2(u)\, dz^2 \tag{2.1}$$

with positive smooth functions a and b. It represents a gravitational wave if

[1] Universität Potsdam, Institut für Mathematik, D-14415 Potsdam, PF 601553, Germany.

0020-7748/98/0300-69 1$15.00/0

$$a \cdot \frac{d^2 b}{du^2} + b \cdot \frac{d^2 a}{du^2} = 0 \tag{2.2}$$

For this result see, e.g., Schimming (1974). Metric (2.1) represents a flat spacetime if both a and b are linear functions.

Let I be any curvature invariant of order k, i.e.,

$$I = I\,(g_{ij}, R_{ijlm}, \ldots, R_{ijlm;\, i_1 \ldots i_k})$$

is a scalar which depends continuously on all its arguments; the domain of dependence is requested to contain the flat space, and $I(g_{ij}, 0, \ldots 0) \equiv 0$.

The following holds (Hawking and Ellis, 1973; Jordan *et al.*, 1960). For gravitational waves of type (2.1), I identically vanishes. Moreover, one can prove that statement for all metrics (2.1) without requiring (2.2). The proof by calculating the components of the curvature tensor is possible, but quite technical.

A very short and geometrical proof goes as follows: Apply a Lorentz boost in the u–v plane, i.e., $u \to \lambda \cdot u$ and $v \to \lambda^{-1} \cdot v$ for any $\lambda > 0$. Then $a(u)$ is replaced by $a(\lambda \cdot u)$ and $b(u)$ by $b(\lambda \cdot u)$. In the limit $\lambda \to 0$, metric (2.1) has a unique limit: constant functions a and b. It is the flat spacetime, and so $I = 0$ there. On the other hand, for all positive values λ, I carries the same value. By continuity this value equals zero. Q.E.D.

Why have all attempts failed to generalize the idea of this proof to the positive-definite case? Because we need a sequence of Lorentz boosts which does not possess any accumulative points within $SO(3, 1)$. Such a sequence does not exist in $SO(4)$, because of compactness.

3. CARTAN SCALARS

A variant (Geroch *et al.*, 1973) of the Newman–Penrose formalism uses projections to an eigennullframe of the curvature tensor to classify gravitational waves. The corresponding Cartan "scalars" have different boost weights, and they represent curvature invariants for vanishing boost weight only. The nonvanishing Cartan "scalars" for metric (2.1) have either nonvanishing boost weight or a discontinuity at flat spacetime. So, the GHP formalism (Geroch *et al.*, 1973; cf. also Dautcourt *et al.*, 1981; Ludwig and Edgar, 1996), does not yield a contradiction to the statement in Section 2.

Why do not there exist analogous "scalars" with different "rotation weight" in the Euclidean signature case? Analyzing the construction, one can see that the boost weights appear because there is a nontrivial vector space isomorphic to a closed subgroup of $SO(3, 1)$. For $SO(4)$, however, it holds that every closed subgroup is compact; in order for it to be isomorphic to a vector space, it is necessary that it be the trivial one-point space.

More geometrically, this looks as follows (Schmidt, 1995). Let $v \in E^4$ be a vector and $g \in SO(4)$ such that $g(v) \uparrow\uparrow v$, then it holds that $g(v) = v$. In Minkowski space-time M^4, however, there exist vectors $v \in M^4$ and boosts $h \in SO(3, 1)$ with $h(v) \uparrow\uparrow v$ and $h(v) \neq v$.

4. LORENTZ-INVARIANT NEIGHBORHOODS

There is no doubt that the Euclidean topology τ is the adequate topology of the Euclidean space E^n. However, controversies appear if one asks whether τ is best suited for the Minkowski spacetime M^n.

The most radical path to answering this question can be found in Hawking *et al.* (1976), Schmidt (1984), and Fullwood (1992); it leads to a topology different from τ which fails to be a normal one.

Here, we only want to find out in which sense one can say that τ is better adapted to E^n than to M^n. On a first view they appear on an equal footing: Both $SO(n)$ and $SO(n - 1, 1)$ represent subgroups of the homeomorphism group of τ.

The difference appears as follows: for E^n, the usual ε-spheres form a neighborhood basis composed of $SO(n)$-invariant open sets. Moreover, each of these neighborhoods has a compact closure. Let U be any open neighborhood with compact closure around the origin in E^n. For every $g \in SO(n)$, $U(g)$ is the set U after rotation by g. Of course, $U(g)$ is also an open neighborhood with compact closure around the origin in E^n. Let us define

$$V = \cup \{U(g) | g \in SO(n)\}$$

and

$$W = \cap \{U(g) | g \in SO(n)\}$$

Both V and W represent $SO(n)$-invariant neighborhoods of the origin with compact closure. Analyzing the proofs, one can see that, "V is a neighborhood of the origin" and "W has compact closure" are trivial statements, whereas "W is a neighborhood of the origin" and "V has compact closure" essentially need the compactness of $SO(n)$. For M^n, however, both of these latter properties fail. In more detail we have:

1. No point of M^n possesses a neighborhood basis composed of $SO(n - 1, 1)$-invariant open sets.

2. No $SO(n - 1, 1)$-invariant neighborhood has a compact closure.

3. Let U be any open neighborhood with compact closure around the origin in M^n. For every $g \in SO(n - 1, 1)$, $U(g)$ is also an open neighborhood with compact closure around the origin in M^n. However, neither

$$V = \cup \{U(g) | g \in SO(n-1, 1)\}$$

nor

$$W = \cap \{(U(g) | g \in SO(n-1, 1)\}$$

represents a neighborhoods of the origin with compact closure.

5. GEODESICS

Now we analyze the following statement (see, e.g., Hawking and Ellis, 1973; Schmidt, 1993): In spite of the fact that the de Sitter spacetime is connected and geodetically complete, there are points in it which cannot be joined to each other by any geodesic.

Let us recall that for Riemannian spaces, if the space is connected and geodetically complete, then each pair of points can be connected by a geodesic.

The proof for Riemannian spaces V_n goes as follows: Take one of its points as x and define $M_x \subset V_n$ to be that set of points which can be reached from x by a geodesic. One can show that M_x is nonempty, open, and closed. This implies $M_x = V_n$.

But where does the corresponding proof fail when we try to generalize it to the de Sitter spacetime?

Let us recall that a geodetic ε-ball is the exponentiated form of a rotation-invariant neighborhood of the corresponding tangent space. For Riemannian spaces these geodetic ε-balls form a neighborhood basis—and just this is needed in the proof.

But where does it fail in detail? M_x "non-empty, open, and closed" would again imply $M_x = V_n$. M_x "nonempty" is trivially satisfied by $x \in M_x$. So we can fail by proving "open" or by proving "closed." It turns out (Schmidt, 1993) that M_x is neither open nor closed in general, and both properties fail by the lack of a neighborhood basis consisting of geodetic ε-balls.

So, if compared with Section 4, we can see that it is again the non-compactness of the Lorentz group which produces the peculiarities.

6. CONCLUSION

A finite set in set theory, a bounded set in geometry, and a compact set in topology: these are corresponding fundamental notions.

What have we learned from the above analysis on compactness? Let us concentrate on the first point (Section 2): the fact that nonisometric spacetimes exist which cannot be distinguished by curvature invariants is neither connected with the fact that one of them is flat nor with the vanishing of the curvature invariants, but, as we have seen, with the appearance of a Lorentz boost which has a limit not belonging to $SO(3, 1)$, but producing a regular metric there. So we have found the very recipe to construct several classes of such spacetimes. Let us present one of them (Schmidt, 1995):

For a positive C^∞-function $a(u)$ let

$$ds^2 = \frac{1}{z^2}[2\, du\, dv - a^2(u)\, dy^2 - dz^2]$$

In the region $z > 0$, ds^2 represents the anti-de Sitter spacetime if and only if $a(u)$ is linear in u. Now, let $d^2a/du^2 < 0$ and

$$\phi := \frac{1}{\sqrt{\kappa}} \int \left(-\frac{1}{a} \frac{d^2 a}{du^2} \right)^{1/2} du$$

Then

$$\Box\phi = \phi_{,i}\, \phi^{,i} = 0 \qquad \text{and} \qquad R_{ij} - \frac{R}{2} g_{ij} = \Lambda g_{ij} + \kappa T_{ij}$$

with $\Lambda = -3$ and $T_{ij} = \phi_{,i}\, \phi_{,j}$. So (ds^2, ϕ) represents a solution of Einstein's equation with negative cosmological term Λ and a minimally coupled massless scalar field ϕ. Let I be a curvature invariant of order k. Then for the metric ds^2, I does not depend on the function $a(u)$. So I takes the same value both for linear and nonlinear functions $a(u)$. This seems to be the first example that nonisometric spacetimes with nonvanishing curvature scalar cannot be distinguished by curvature invariants. And having the recipe, the construction of other classes is straightforwardly done.

The fact that the representation theory of the rotation groups $SO(n)$ and the Lorentz groups $SO(n - 1, 1)$ is quite different is so well known that we did not repeat it here—we only want to mention that it is the compactness of the first one which produces the difference.

ACKNOWLEDGMENTS

Financial support from Deutsche Forschungsgemeinschaft, and valuable comments by Alan Held and Martin Rainer, are gratefully acknowledged.

REFERENCES

Dautcourt, G., Jann, K.-P., Riemer, E., and Riemer, M. (1981). *Astronomische Nachrichten,* **302**, 1.

Fullwood, D. (1992). *Journal of Mathematical Physics,* **33**, 2232.

Geroch, R., Held, A., and Penrose, R. (1973). *Journal of Mathematical Physics,* **14**, 874.

Hawking, S., and Ellis, G. F. R. (1973). *The Large Scale Structure of Space-Time,* Cambridge University Press, Cambridge.

Hawking, S., King, A., and McCarthy, P. (1976). *Journal of Mathematical Physics,* **17**, 174.

Jordan, P., Ehlers, J., and Kundt, W. (1960). *Abhandlungen der Akademie der Wissenschaften, Mainz. Mathematik/Naturwissenschaften,* 21.

Ludwig, G., and Edgar, S. (1996). *General Relativity and Gravitation,* **28**, 707.

Schimming, R. (1974). *Mathematische Nachrichten,* **59**, 129.

Schmidt, H.-J. (1984). *Abhandlungen der Akademie der Wissenschaften der DDR, Abteilung Mathematik,* **2N**, 207.

Schmidt, H.-J. (1993). *Fortschritte der Physik,* **41**, 179.

Schmidt, H.-J. (1995). In *Abstracts GR 14 Florence,* A61.

Schmidt, H.-J. (1996). In *New Frontiers in Gravitation,* G. Sardanashvily, ed., Hadronic Press, Palm Harbor, Florida, p. 337.